

Numerical Solution of Differential Equations in Irregular Plane Regions Using Quality Structured Convex Grids

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Abstract

The variational grid generation method is a powerful tool for generating structured convex grids on irregular simply connected domains whose boundary is a polygonal Jordan curve. Several examples that show the accuracy of a difference approximation to the solution of a Poisson equation using these kind of structured grids have been recently reported. In this paper, we compare the accuracy of the numerical solution calculated by applying those structured grids with finite differences against the the solution obtained with Delaunay-like triangulations on irregular regions.

1 Introduction

For the numerical solution of the Poisson equation on irregular domains, the use of finite differences and finite elements with triangulations obtained by subdividing each grid cell of a structured grid along a diagonal has been addressed in Ref. [8], proving that the finite difference approach on such grids is accurate enough. However, since structured grids often have some elongated cells, a natural question that arises in this context is whether this numerical solution is more accurate than the solution obtained by using finite elements on a standard Delaunay-like triangulation.

In this paper, we compare the accuracy of the numerical solution for this problem using finite differences in structured grids, and finite elements on Delaunay-like triangulations. We are specifically interested in irregular boundaries, since their geometry is closer to a realistic domain, for instance, a lake.

In order to generate the structured convex grids, a variational method was used. The latter consists of minimizing an appropriate functional [6, 9, 10, 13]. Area and harmonic functionals can be used for gridding a wide variety of simple connected domains in the plane [2, 3, 4, 11, 16, 17], whose boundaries are closed polygonal Jordan curves with positive orientation.

If m and n represent the “vertical” and “horizontal” numbers of points of the “sides”, then the boundary is the positively oriented polygonal curve γ of vertices $V = \{v_1, \dots, v_{2(m+n-2)}\}$, and it defines the typical domain Ω .

A doubly indexed set

$$G = \{P_{i,j} | 1 \leq i \leq m, 1 \leq j \leq n\}$$

of points of the plane with the fixed boundary positions given by V is a logically rectangular structured grid with quadrilateral elements for Ω , of order $m \times n$.

A grid G is convex if and only if each one of the $(m-1)(n-1)$ quadrilaterals (or cells) $c_{i,j}$ of vertices $\{P_{i,j}, P_{i+1,j}, P_{i,j+1}, P_{i+1,j+1}\}$, $1 \leq i < m$, $1 \leq j < n$, is convex and non-degenerate (See fig. 1).

The basis for the direct optimization method, as developed by Ivanenko *et al.*

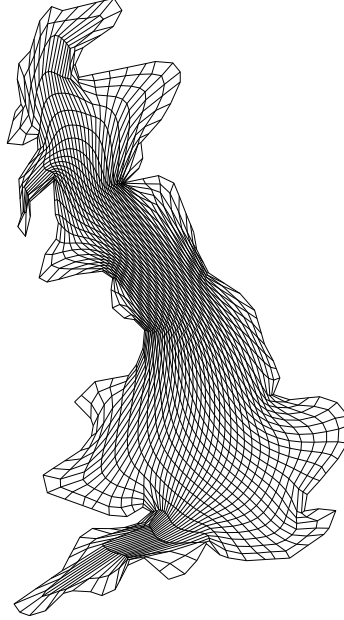


Figure 1: Structured grid generated by UNAMALLA

[9], is the minimization a suitable function of the form

$$F(G) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f(c_{i,j}), \quad (1)$$

where $c_{i,j}$ is the $(i,j)^{th}$ grid cell and f is a function of its vertices; the problem is to find the coordinates of the interior points of the grid G . The functional used to generate the structured grids of the numerical tests, as implemented in UNAMALLA [18], was the adaptive linear convex combination of the area functional $S_\omega(G)$ described in Ref. [5], and the length functional $L(G)$ with weight $\sigma = 0.5$ (See Ref. [2]).

The parameter ω , which is a scale factor, can be updated in such a way that in a finite number of updates the combined functional attain its minima within the set of convex grids for Ω if the latter is nonempty. Further properties of this functional, as well as the algorithm for updating its parameter has been reported in Ref. [2] and Ref. [5].

The triangulations we considered for this paper are those generated by DistMesh [12], which is based on the physical analogy between a simplex mesh and a truss structure, where the meshpoints are nodes of the truss. It generates an initial Delaunay triangulation, then assumes an appropriate force-displacement function for the bars in the truss at each iteration, and finally solves for equilibrium (See fig. 2). In order to have grids with a similar number of elements, we used two initializations: a) the initial edge length for DistMesh was set in proportion to half the average diagonal length of the structured grids, b) a variation of DistMesh was designed for which the initial points inside the region are the inner nodes of the corresponding structured grid.

It must be noted that for very irregular boundaries, DistMesh might produce a few triangular elements along the boundary which do not satisfy the Delaunay condition.

2 Finite difference approximation

Standard difference schemes can be generalized by considering a finite set of nodes p_1, p_2, \dots, p_k , for which it is required to find coefficients $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ such that [7]

$$\frac{\partial^q u}{\partial x^l \partial y^{q-l}}|_{x=x_*} \approx \sum_i \Gamma_i u(p_i). \quad (2)$$

As it is well known, the Γ values can be calculated with ease in regular regions. However, despite the basic idea is quite simple, the application to Taylor's Theorem leads to more complicated schemes on irregular regions; up to our knowledge, there are few efficient schemes for such kind of regions.

The calculation of these coefficients has been studied by Tinoco *et al* [1], and Shashkov [14]. We make use of the second order scheme developed for the



Figure 2: Triangulation generated by DistMesh

boundary value problem

$$\begin{aligned}
 -\nabla(K(x,y)\nabla u(x,y)) &= f(x,y) \\
 K(x,y) &= \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} \\
 u(x,y)|_{\partial\Omega} &= g(x,y),
 \end{aligned} \tag{3}$$

with non-diagonal matrices $K(x,y)$ studied in Ref. [14], which is based on the method of support-operators and has the advantage of providing explicit formulas for the Γ coefficients¹.

In the numerical experiments, we selected the 3×3 subgrid defined for the nodes $x(i-1:i+1, j-1:j+1)$, $y(i-1:i+1, j-1:j+1)$ to approximate $-\nabla(K(x,y)\nabla u(x,y)) = f(x,y)$ at the inner grid node $(x(i,j), y(i,j))$. As in the rectangular case, an algebraic system of equations is obtained from discretization, which becomes sparse as m and n increase.

¹This scheme is second order according to the grid norm defined by equation (7).

3 Finite elements approximation

Let Ne be the number of triangular elements in a grid. Galerkin's approach to the solution of equation (3) is given by the combination

$$u(x, y) \approx \sum_{i=1}^{Ne} U_i \phi_i \quad (4)$$

of trial-test pyramid functions whose faces are defined on a triangle by

$$\phi(x, y) = A + Bx + Cy. \quad (5)$$

This selection yields the weak formulation

$$\sum_{i=1}^{Ne} U_i \int_{\Omega} \langle \nabla \phi_j, K(x, y) \nabla \phi_i \rangle d\mathcal{A} = \int_{\Omega} \phi_j f d\mathcal{A}, \quad j = 1, \dots, Ne, \quad (6)$$

where $\langle \cdot, \cdot \rangle$ is the canonical inner product (See Ref. [15]).

There are several other possible choices of elements. However, as it is well known, the use of pyramids on triangulations allows a very simple second order approximation to the solution of equation (3).

4 Numerical tests

For the numerical tests, we have selected 9 polygonal regions, most of them approximations to real geographical locations: they will be denoted as Dome (dom), Great Britain (eng), Havana bay (hab), M19 (m19), México (mex), Plow (plo), Swan (swa), Ucha (uch) and Michoacán (mic). They are shown in figure 3.

Scaling these boundaries in order to lie in $[0, 1] \times [0, 1]$, convex grids with 21, 41 and 81 points per side were generated by minimizing the $1/2(S_{\omega}(G) + L(G))$ functional in UNAMALLA with default parameters. The resulting structured grids were used with Shashkov's finite difference schemes [14]. As mentioned before, they were also used as initial data for some triangulations.

DistMesh was used to triangulate the same test polygonal boundaries with default parameters, setting the bar length as half the average diagonal length calculated in the corresponding structured grids (denoted as *DistMesh_a*). A variation of DistMesh was also designed, for which the initial points inside the region are the inner nodes of the corresponding structured grid: these grids are denoted as *DistMesh_b* grids.

The number of elements and inner nodes in each grid can be seen in in table 1. The column N gives twice the number of grid cells², and the columns N_a

²This number is considered because the number of triangles in a triangulation obtained by subdividing each grid cell along a diagonal of a structured grid is twice the number of cells.

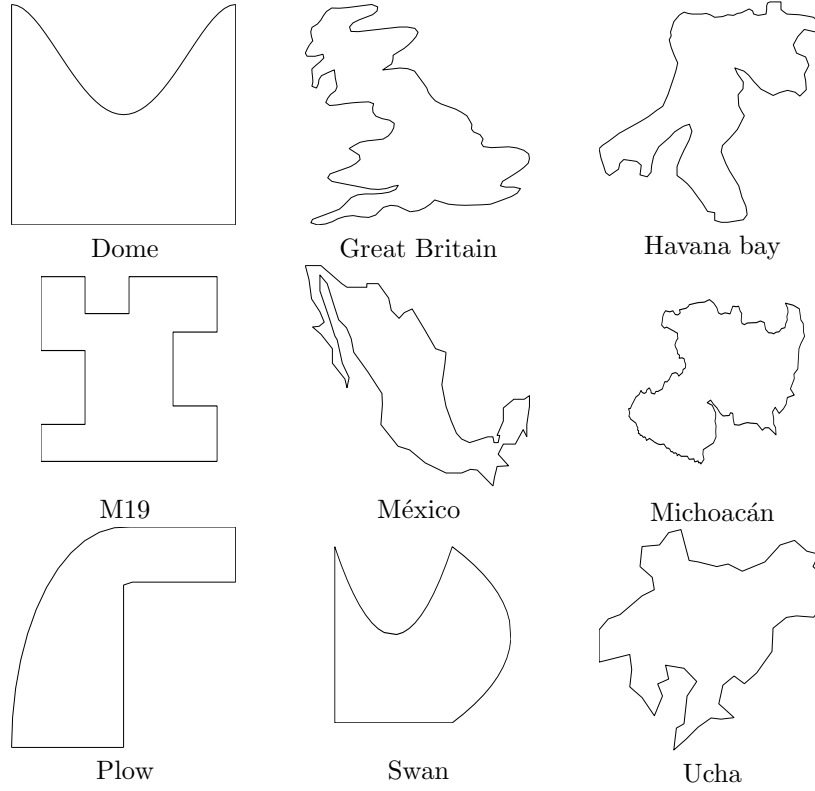


Figure 3: Test regions.

and N_b the number of triangular elements in the $DistMesh_a$ and $DistMesh_b$ triangulations, respectively. The columns labeled Nu , Nu_a and Nu_b represent the corresponding number of inner nodes, which is equal to the number of unknowns in the approximation.

The resulting algebraic systems for finite differences and elements are sparse due to the discretization. However, the systems for finite differences are block-tridiagonal, so they can be solved with a number of algorithms in a very efficient way; this is a clear advantage of the double index in a structured grid. For the tests, Gauss-Seidel Method was used.

On the other hand, the matrices in the finite element systems have no specific structure and require a more complicated data structure in order to solve them efficiently. For the tests, they were solved using a sparse Gaussian elimination routine.

The following values for u and K were selected (See Ref. [14]):

Table 1: Number of grid elements and inner nodes

Region	Size	N_1	N_2	N_3	Nu	Nu_a	Nu_b
dom	21	800	1540	795	361	693	356
	41	3200	6064	3189	1521	2884	1509
	81	12800	24150	12783	6241	11785	6218
eng	21	800	841	771	361	355	337
	41	3200	3463	3111	1521	1581	1440
	81	12800	13918	12614	6241	6622	6074
hab	21	800	1261	767	361	559	333
	41	3200	5048	3115	1521	2370	1446
	81	12800	20313	12605	6241	9845	6070
m19	21	800	1420	785	361	626	350
	41	3200	5717	3157	1521	2696	1483
	81	12800	22772	12705	6241	11067	6156
mex	21	800	477	686	361	183	273
	41	3200	1781	2911	1521	771	1288
	81	12800	7036	11918	6241	3301	5519
mic	21	800	1432	796	361	645	356
	41	3200	5760	3183	1521	2747	1506
	81	12800	22812	12694	6241	11096	6136
plo	21	800	863	748	361	363	310
	41	3200	3335	3076	1521	1535	1399
	81	12800	13115	12537	6241	6303	5983
swa	21	800	1410	796	361	634	357
	41	3200	5495	3187	1521	2612	1508
	81	12800	21737	12766	6241	10580	6205
uch	21	800	1088	794	361	483	355
	41	3200	4252	3166	1521	1976	1489
	81	12800	17160	12729	6241	8288	6172

1. First problem.

$$K(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad u = 2 \exp(2x + y).$$

2. Second problem.

$$K(x, y) = P^T D P,$$

with

$$P = \begin{pmatrix} \cos(\frac{\pi}{8}) & \sin(\frac{\pi}{8}) \\ -\sin(\frac{\pi}{8}) & \cos(\frac{\pi}{8}) \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 + 2x^2 + y^2 & 0 \\ 0 & 1 + x^2 + 2y^2 \end{pmatrix}$$

$$u = \sin(\pi x) \sin(\pi y).$$

3. Third problem.

$$K(x, y) = P^T D P,$$

with

$$P = \begin{pmatrix} \cos(\frac{\pi}{4}) & \sin(\frac{\pi}{4}) \\ -\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 + 2x^2 + y^2 + y^5 & 0 \\ 0 & 1 + x^2 + 2y^2 + x^3 \end{pmatrix}$$

$$u = \sin(\pi x) \sin(\pi y).$$

Function f was chosen in such a way that u was the exact solution in every case.

The $\|\cdot\|_2$ error norm for the tests is summarized in tables 2, 3 and 4; it was calculated as a grid function

$$\|u - U\|_2 = \sqrt{\sum_i (u_i - U_i)^2 \mathcal{A}_i} \quad , \quad (7)$$

where u and U are the approximated and the exact solution calculated at the i^{th} -element, and \mathcal{A}_i is the area of the element. The approximation corresponding to label *Structured* are those of the second order finite differences mentioned in section 2; the approximations for the columns labeled *DistMesh_a*, and *DistMesh_b* were calculated with finite elements using the pyramid trial-test approximation described in section 3. The empirical orders O , O_a and O_b between two consecutive grid orders were calculated according to the formula

$$\log \left(\frac{E_i}{E_j} \right) / \log \left(\frac{n_j}{n_i} \right) \quad (8)$$

Table 2: Quadratic error for problem 1

Region	Size	<i>Structured</i>	<i>O</i>	<i>DistMesh_a</i>	<i>O₁</i>	<i>DistMesh_b</i>	<i>O₂</i>
dom	21	4.59E-03		5.77E-03		6.25E-03	
	41	1.22E-03	1.98	8.79E-04	2.81	1.47E-03	2.16
	81	1.46E-04	3.12	1.51E-04	2.58	2.70E-04	2.49
eng	21	2.58E-03		1.72E-03		2.44E-03	
	41	4.27E-04	2.69	6.95E-05	4.79	4.75E-04	2.45
	81	1.17E-04	1.90	1.03E-04	-0.57	2.86E-04	0.75
hab	21	7.53E-03		1.26E-03		1.98E-03	
	41	1.53E-03	2.38	1.69E-04	3.00	5.85E-04	1.82
	81	3.97E-04	1.99	6.11E-05	1.50	1.77E-04	1.76
m19	21	5.90E-03		2.02E-03		1.94E-02	
	41	1.44E-03	2.11	6.55E-04	1.68	7.18E-04	4.93
	81	3.21E-04	2.20	8.24E-04	-0.34	1.39E-04	2.41
mex	21	8.67E-03		2.06E-03		8.03E-04	
	41	1.87E-03	2.29	2.34E-04	3.25	3.72E-04	1.15
	81	3.85E-04	2.32	4.93E-05	2.29	3.82E-05	3.34
mic	21	5.53E-03		1.45E-04		8.15E-04	
	41	1.77E-03	1.70	7.00E-05	1.09	9.21E-04	-0.18
	81	3.73E-04	2.29	3.45E-04	-2.34	4.61E-05	4.40
plo	21	1.04E-03		1.52E-03		4.49E-04	
	41	5.06E-04	1.08	6.05E-04	1.38	1.00E-04	2.24
	81	7.97E-05	2.71	1.77E-04	1.81	2.80E-05	1.87
swa	21	2.57E-03		1.08E-03		1.77E-03	
	41	6.01E-04	2.17	3.73E-04	1.58	4.54E-04	2.03
	81	1.50E-04	2.04	5.45E-05	2.82	9.28E-05	2.33
uch	21	1.14E-02		3.14E-03		3.05E-03	
	41	1.91E-03	2.67	1.08E-04	5.04	7.05E-04	2.19
	81	3.78E-04	2.38	7.68E-04	-2.88	1.84E-04	1.97

Table 3: Quadratic error for problem 2

Region	Size	<i>Structured</i>	<i>O</i>	<i>DistMesh_a</i>	<i>O₁</i>	<i>DistMesh_b</i>	<i>O₂</i>
dom	21	9.68E-04		4.21E-04		5.40E-04	
	41	2.26E-04	2.18	7.54E-05	2.57	1.44E-04	1.98
	81	4.75E-05	2.29	1.04E-05	2.91	3.16E-05	2.23
eng	21	9.72E-04		6.31E-04		7.68E-04	
	41	1.83E-04	2.50	2.49E-05	4.83	2.63E-04	1.60
	81	5.00E-05	1.90	4.40E-05	-0.84	1.30E-04	1.04
hab	21	1.08E-03		1.95E-04		3.54E-04	
	41	3.90E-04	1.52	2.79E-05	2.91	6.58E-05	2.51
	81	9.72E-05	2.04	1.37E-05	1.04	1.99E-05	1.76
m19	21	1.30E-03		2.93E-04		2.34E-03	
	41	2.75E-04	2.32	2.52E-05	3.66	1.26E-04	4.37
	81	8.46E-05	1.73	1.33E-04	-2.45	2.91E-05	2.15
mex	21	1.96E-03		5.39E-04		3.24E-04	
	41	3.00E-04	2.80	8.17E-05	2.82	9.53E-05	1.83
	81	7.54E-05	2.03	1.46E-05	2.53	4.65E-06	4.44
mic	21	8.01E-04		2.78E-05		1.08E-04	
	41	2.03E-04	2.05	5.99E-05	-1.15	7.51E-05	0.55
	81	6.72E-05	1.62	3.90E-05	0.63	4.65E-06	4.09
plo	21	2.36E-04		2.87E-04		1.13E-04	
	41	9.72E-05	1.33	1.18E-04	1.33	2.26E-05	2.41
	81	2.28E-05	2.13	2.31E-05	2.40	4.98E-06	2.22
swa	21	4.63E-04		2.35E-04		3.66E-04	
	41	1.14E-04	2.10	7.44E-05	1.72	8.81E-05	2.13
	81	3.37E-05	1.79	1.15E-05	2.75	2.04E-05	2.15
uch	21	1.33E-03		9.61E-04		4.75E-04	
	41	2.59E-04	2.44	2.29E-05	5.59	1.30E-04	1.94
	81	6.14E-05	2.12	1.33E-04	-2.58	3.00E-05	2.15

Table 4: Quadratic error for problem 3

Region	Size	<i>Structured</i>	O	$DistMesh_a$	O_1	$DistMesh_b$	O_2
dom	21	9.70E-04		4.22E-04		5.49E-04	
	41	2.26E-04	2.18	7.55E-05	2.57	1.47E-04	1.96
	81	4.74E-05	2.29	1.04E-05	2.91	3.15E-05	2.27
eng	21	9.69E-04		6.33E-04		6.94E-04	
	41	1.81E-04	2.50	2.47E-05	4.85	2.60E-04	1.47
	81	4.98E-05	1.90	4.43E-05	-0.85	1.37E-04	0.94
hab	21	1.05E-03		1.96E-04		3.55E-04	
	41	3.84E-04	1.51	2.76E-05	2.93	6.57E-05	2.52
	81	9.54E-05	2.05	1.37E-05	1.03	2.00E-05	1.75
m19	21	1.29E-03		2.91E-04		2.26E-03	
	41	2.72E-04	2.33	2.46E-05	3.70	1.24E-04	4.34
	81	8.38E-05	1.73	1.36E-04	-2.52	2.87E-05	2.15
mex	21	2.01E-03		5.55E-04		3.18E-04	
	41	3.04E-04	2.82	8.10E-05	2.88	9.42E-05	1.82
	81	7.32E-05	2.09	1.45E-05	2.53	4.65E-06	4.42
mic	21	8.13E-04		2.81E-05		1.10E-04	
	41	2.03E-04	2.07	7.72E-05	-1.51	8.61E-05	0.37
	81	6.77E-05	1.61	4.39E-05	0.83	4.96E-06	4.19
plo	21	2.36E-04		3.05E-04		1.14E-04	
	41	9.84E-05	1.31	1.23E-04	1.36	2.28E-05	2.41
	81	2.31E-05	2.13	2.30E-05	2.46	5.04E-06	2.22
swa	21	4.44E-04		2.44E-04		3.93E-04	
	41	1.07E-04	2.12	7.82E-05	1.70	9.46E-05	2.13
	81	3.32E-05	1.72	1.21E-05	2.74	2.18E-05	2.15
uch	21	1.32E-03		1.01E-03		4.97E-04	
	41	2.57E-04	2.45	2.29E-05	5.66	1.35E-04	1.95
	81	6.26E-05	2.07	1.35E-04	-2.60	3.09E-05	2.17

where E_i is the quadratic error associated to the numerical solution calculated with a grid with n_i points per side.

The quadratic error for the grids with 81 points per side for the three problems is sketched in figures 4, 5 and 6.

Some conclusions can be drawn from the numerical results. The strong non convexities on some boundaries are clearly reflected in the error magnitudes and in a slight decrease of the empirical order; this can be explained in terms of the presence of elongated elements. For instance, in the *DistMesh_a* grids, the boundary triangles must preserve the boundary shape, so the boundary nodes are kept fixed (see fig. 2) and in consequence some elongated elements are produced. Nevertheless, one must also note that the very elongated cells in the structured grids have less negative effect in the solution than expected.

It can also be seen that in a number of problems the solution calculated with finite elements is slightly more accurate than than solution obtained with finite differences. But the former often required a larger number of unknowns, which increases the computational effort required for the calculations.

However, at the end, one conclusion arises: in the experiments, no method seems to be clearly superior. The numerical solutions obtained by second order finite differences in the structured grids generated by variational methods are essentially as accurate as that obtained by second order finite elements using Delaunay-like triangulations. Having in mind this fact, an additional important issue must be discussed: the simplicity of use of the data structures required for finite differences. Since structured grids are logically rectangular, equation (2) is algorithmically as simple as a nine point discrete laplacian. This is an advantage over the more complicated data structures required for triangulations, since the numerical results do not show a significative improvement.

5 Conclusions and future work

In this paper, we selected irregular planar regions instead of rectangular ones, which are often a poor approximation to real-world domains. In addition, the side selection was quite arbitrary, and no boundary was processed in any way in order to avoid simplifying the problems. Thus, strong non convex boundaries are clearly reflected in the quadratic errors (*e.g. Great Britain, M19 and Ucha*), but these choices were done so because we wanted to deal with a “hard” problem, although it must be acknowledged that boundary preprocessing is an excellent strategy to improve numerical results in general grids.

However, the use of irregular fixed boundaries is precisely what leads to the main conclusion: as follows from the numerical experiments, no method seems to be notably more accurate than the others; difference schemes applied on structured meshes can indeed be used to produce reliable approximations to the solution of the test problems. In other words, triangulations are not the only reliable choice for such regions, it is also possible to generate accurate numerical

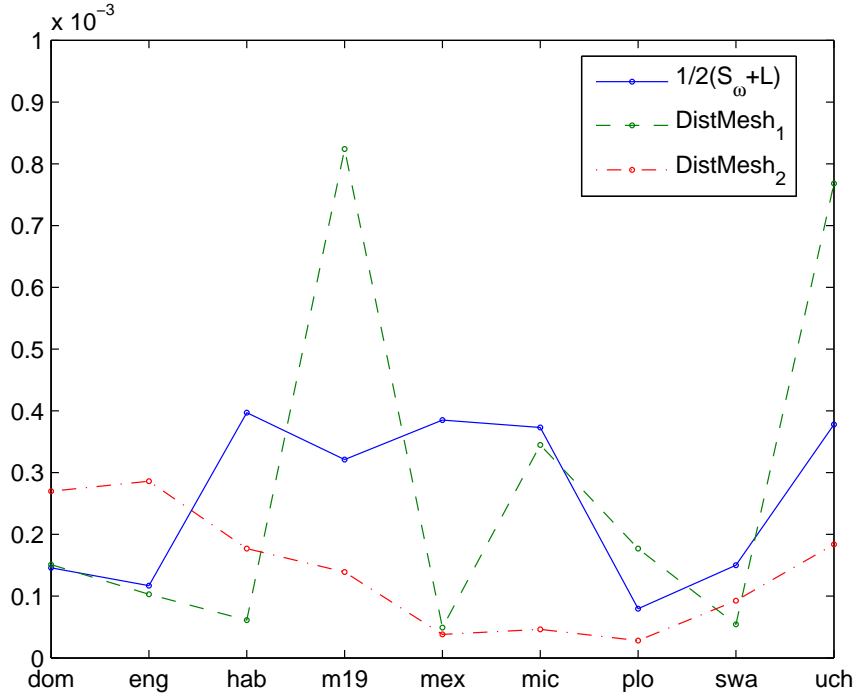


Figure 4: quadratic error for problem 1

solutions using grids generated by variational methods in very irregular regions, and this approach has not been thoroughly studied yet. As we mentioned, this is an important issue, since differences are based on logically rectangular grids, and, in consequence, their algorithmic implementations are rather simple. Our current research deals with time-dependent partial differential equations on irregular domains, and the corresponding results will be reported in a future paper.

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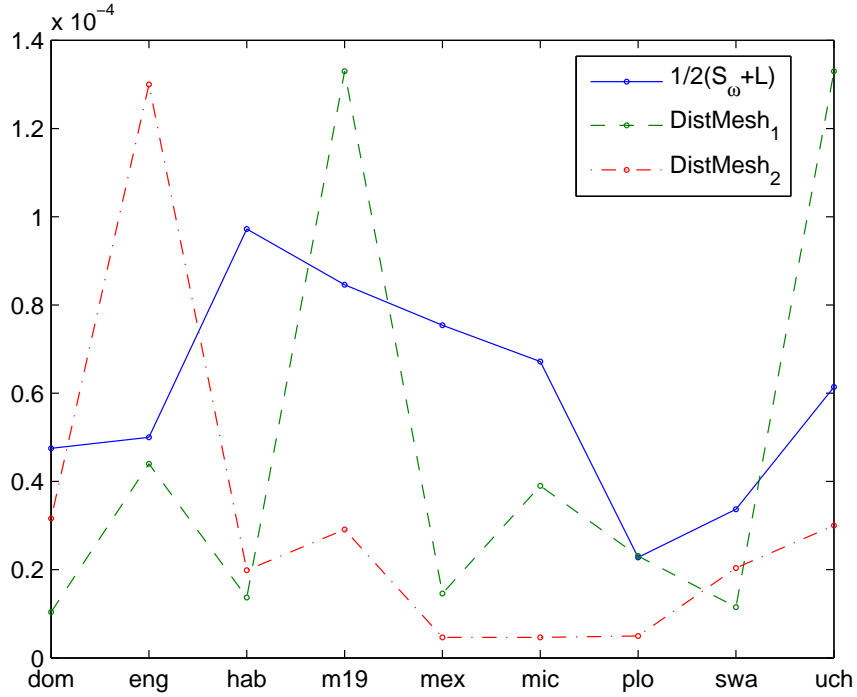


Figure 5: quadratic error for problem 2

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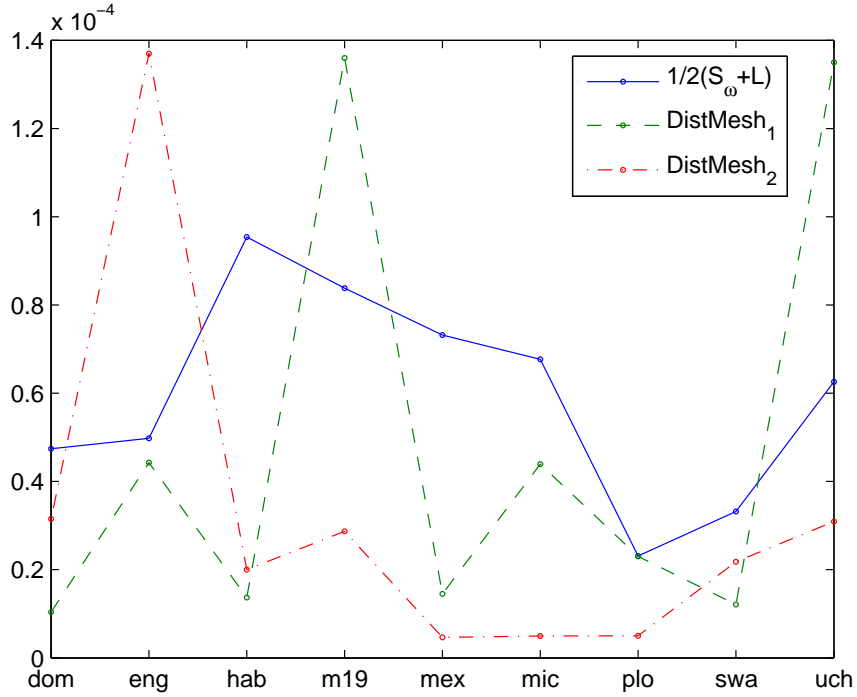


Figure 6: quadratic error for problem 3

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